Linear Algebra II

02/03/2020, Monday, 19:00 – 21:00

1 $(6+7+7=20 \text{ pts})$ **Inner product spaces**

Consider the vector space $\mathbb{R}^{2\times 2}$. Let S be the linear subspace of $\mathbb{R}^{2\times 2}$ spanned by the matrices M_1 and M_2 given by

$$M_1 := \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad M_2 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

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(a) Show that $\langle A, B\rangle := \operatorname{trace}(A^T B)$ defines an inner product on $\mathbb{R}^{2\times 2}$.

(b) Determine an orthonormal basis for the subspace S with respect to this inner product

(c) Determine the best approximation of the matrix $M := \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in the subspace S. Here, a, b, c and d are given real numbers.

2 $(8+4+4+4=20 \text{ pts})$ **Least squares approximation**

For a given positive integer n, let P_n be the vector space of all real polynomials $p(x)$ with degree less than or equal to $n-1$, i.e.

$$P_n = \{p(x) \mid p(x) = p_0 + p_1 x + p_2 x^2 + \ldots + p_{n-1}x^{n-1},\ p_i \in \mathbb{R}\}$$

Let $c_1, c_2, \ldots c_n$ be n distinct real numbers. For any $p(x) \in P_n$ define $\tilde{p} \in \mathbb{R}^n$ by

$$\tilde{p} := \begin{pmatrix} p(c_1) \\ p(c_2) \\ \vdots \\ p(c_n) \end{pmatrix}$$

(a) For any pair of polynomials $p(x), q(x) \in P_n$, define $\langle p(x), q(x)\rangle := \tilde{p}^T\tilde{q}$. prove that $\langle \cdot, \cdot \rangle$ defines an inner product on P_n.

In the remainder of this problem, take $n = 3$ and $c_1 = -1, c_2 = 0, c_3 = 1$.

(b) Show that the polynomial 1 and x are orthogonal.

(c) Determine an orthonormal basis of the subspace P_2 of P_3.

(d) Compute the orthogonal projection $p(x)$ of the polynomial x^2 onto the subspace P_2.

3 $(5 + 10 + 10 = 25 \text{ pts})$ **Diagonalization**

Let A be a real $n \times n$ matrix. Define a new matrix M by $M = A + iA^T$

(a) Show that $M^H = -iM$

(b) Prove that M is unitarily diagonalizable

(c) Show that the eigenvalues of M lie on the line $\{x + ix \mid x \in \mathbb{R}\}$

4 $(3 + 8 + 8 + 6 = 25 \text{ pts})$ **Hermitian matrices**

Let $A \in \mathbb{C}^{n \times n}$ and let Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $A^H A$.

(a) Show that $A^H A$ is Hermitian.

(b) Show that $A^H A$ and AA^H have the same nonzero eigenvalues.

(c) Show that $\lambda_i \geqslant 0$ for all i.

(d) Show that if A has linearly independent columns then $\lambda_i > 0$ for all i.

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10 pts free